Limits

Calculus I

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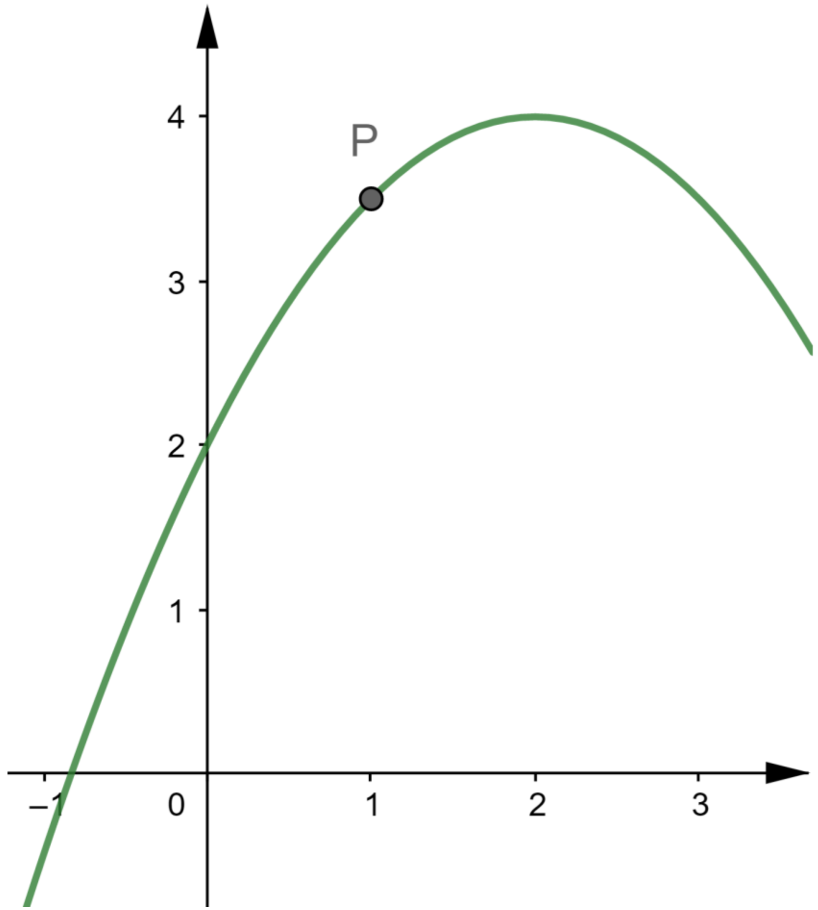
Limits

Motivation:
Looking for a properties at some points



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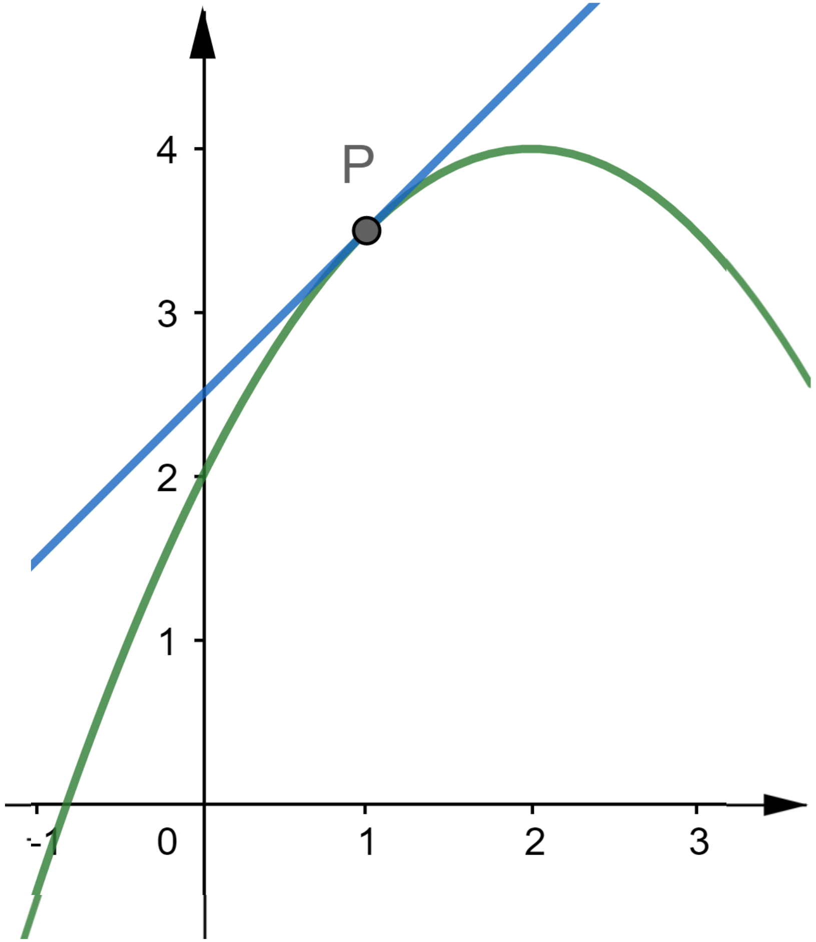

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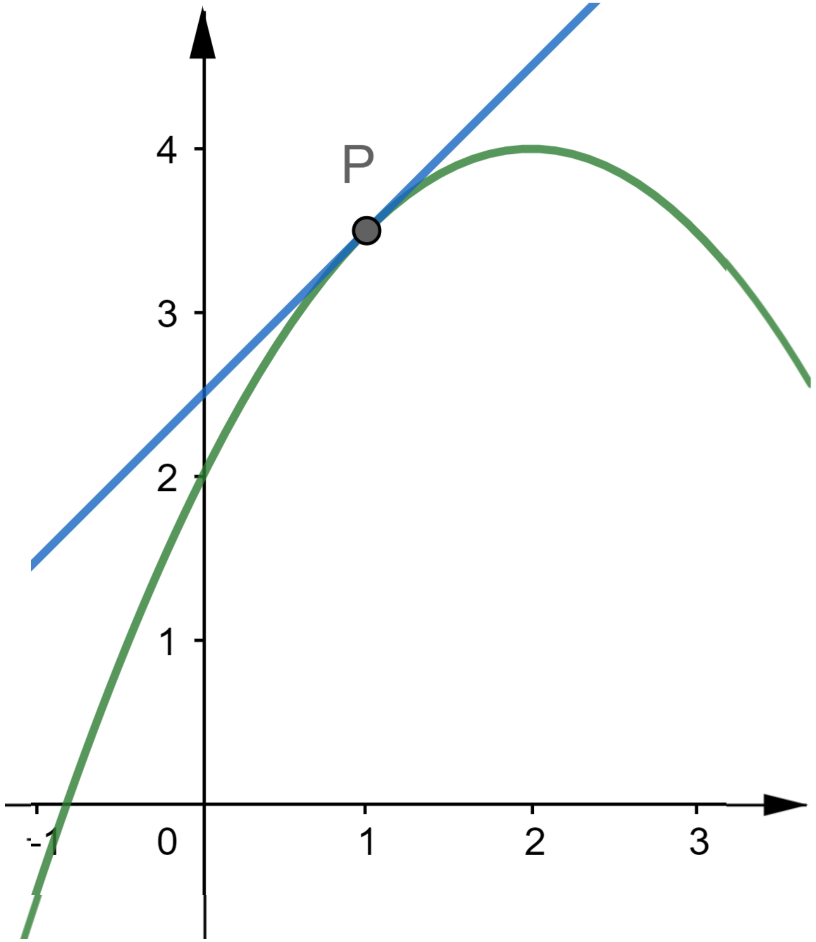

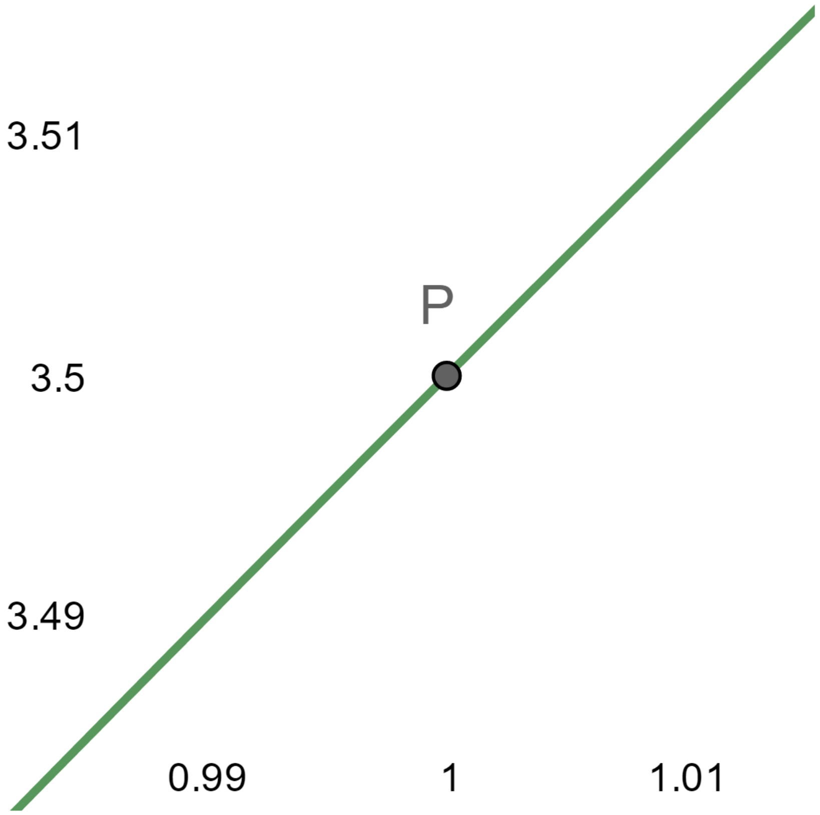

Motivation:
Looking for a pattern

Long journey to find π

- Approximation π with 22/7 or 3.14 for which both are wrong
- Facts:

$$3.14 < \pi < 22/7$$

- What is the exact value of π?

Motivation:
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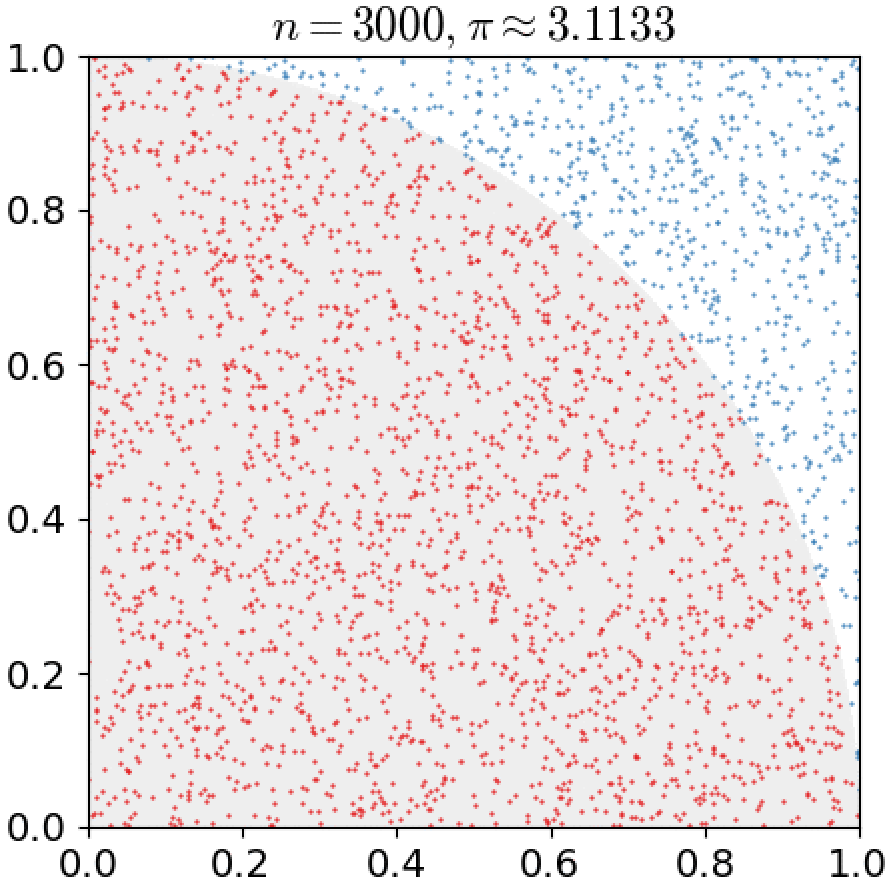

Motivation:
Looking for a pattern

- Series that "converge" to π:

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)$$

n	10	100	1000	10000	100000	1000000
π	3.04183961	3.13159290	3.14059265	3.14149265	3.14158265	3.14159165

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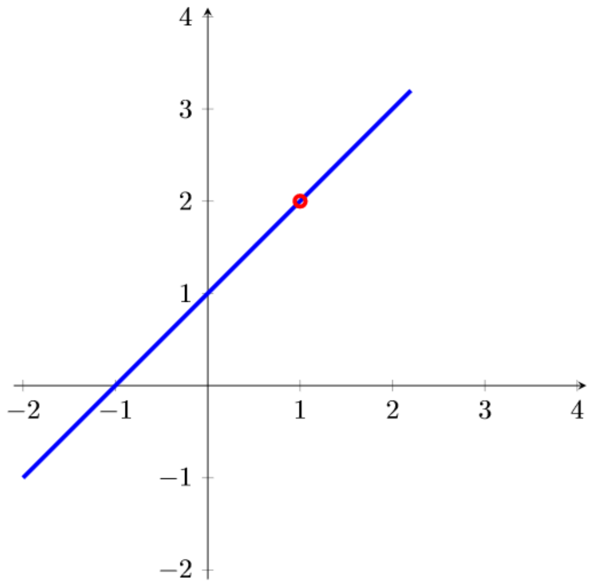
Motivation: Numerical Approximation

- Let f be function

$$f(x) = \frac{x^2-1}{x-1}$$



- At x = 1, we got a problem, $\frac{0}{0}$ is not defined.
- But looking in the neighbourhood we got the following tables.

x	0.8	0.9	0.95	0.99	0.995	0.999	1	1.001	1.005	1.01	1.05	1.1	1.2
f(x)	1.8	1.9	1.95	1.99	1.995	1.999	??	2.001	2.005	2.01	2.05	2.1	2.2

Definition

Limits

Definition

- Let f be function defined on an open interval contains *c,* except possibly at c.
- If the value of $\mathrm{f}(x)$ get closer to L as x get closer to c we say that f has a limit at c.
- We may write $f(x) \to L$ as $x \to c$.

Definition

- Let f be function defined on an open interval contains *c,* except possibly at c.
- If the value of $\mathrm{f}(x)$ get closer to L as x get closer to c we say that f has a limit at c.
- We may write $f(x) \to L$ as $x \to c$.
- Formally, we write it

$$\lim_{x \to c} f(x) = L$$

Formal Definition

Let f be function defined on an open interval contains *c,* except possibly at c. The limit of $f(x)$ as x approach *c* is said to be equals to L, denoted by

$$\lim_{x \to c} f(x) = L$$

if for every $\epsilon > 0$, there exist $\delta > 0$ so that every $x \in D_f$ this implication holds:

$$0 < |x - c| < \delta \Rightarrow 0 < |f(x) - L| < \epsilon$$

Example

Prove that function $f(x) = \frac{x^2-1}{x-1}$ has a limit at x=1, that is 2.

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Proof. Given any $\epsilon > 0$ take $\delta = \epsilon$. Then, if we have $0 < |x - 1| < \delta$, one could find that

$$|f(x) - 2| = \left|\frac{x^2 - 1}{x - 1} - 2\right| = \left|\frac{(x + 1)(x - 1)}{x - 1} - 2\right| = |x + 1 - 2| = |x - 1| < \delta = \epsilon$$

Or simply, $|f(x) - 2| < \epsilon$.

Properties

Limits

Properties: Uniqueness

Theorem. If limit of f exists on *x*=*c*, then it is unique.

In that case, we may write $\lim_{x \to c} f(x)$ as the value of the limits.

Properties

Theorem. Let c and k be real numbers. We have the following properties.

$$\lim_{x \to c} f(k) = k$$

$$\lim_{x \to c} f(x) = c$$

Properties

Theorem. Let *f* and *g* be functions; *k* and *c* be real number. Assume that the limit of f and g at c exists, then we have the following properties.

$$\lim_{x \to c} k\ f(x) + g(x)) = k \lim_{x \to c} f(x)$$

$$\lim_{x \to c} (f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$\lim_{x \to c} (f(x) g(x)) = \left(\lim_{x \to c} f(x)\right)\left(\lim_{x \to c} g(x)\right)$$

$$\lim_{x \to c} (f(x)/g(x)) = \left(\lim_{x \to c} f(x)\right) / \left(\lim_{x \to c} g(x)\right) \quad \text{if } \lim_{x \to c} g(x) \neq 0$$

Properties

Corollary. Let *f* be function; *n* be a natural number. Assume that the limit of f at c exists, then we have the following properties.

$$\lim_{x\to c}\big(f(x)\big)^n = \Big(\lim_{x\to c} f(x)\Big)^n$$

$$\lim_{x\to c}\big(f(x)\big)^{-n} = \Big(\lim_{x\to c} f(x)\Big)^{-n}$$ if $\lim_{x\to c} f(x) \neq 0$

$$\lim_{x\to c}\big(f(x)\big)^{1/n} = \Big(\lim_{x\to c} f(x)\Big)^{1/n}$$ if $\lim_{x\to c} f(x) \geq 0$ whenever *n* is even

Example (1)

$$\lim_{x \to 1} 2x + 1 =$$

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$$\lim_{x \to 1} 2x + 1 = \lim_{x \to 1} 2x + \lim_{x \to 1} 1 = 2 \lim_{x \to 1} x + \lim_{x \to 1} 1 = 2.1 + 1 = 3$$

Example (2)

$$\lim_{x \to 1} \frac{x^2 - 1}{x + 1} =$$

Example (2a)

$$\lim_{x \to 1} \frac{x^2 - 1}{x + 1} = \frac{\left(\lim\limits_{x \to 1} x\right)^2 - 1}{\lim\limits_{x \to 1} x + \lim\limits_{x \to 1} 1} =$$

Example (2)

$$\lim_{x \to 1} \frac{x^2 - 1}{x + 1} =$$

Example (2b)

$$\lim_{x \to 1} \frac{x^2 - 1}{x + 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x + 1} = \lim_{x \to 1} x - 1 = 0$$

Example (3)

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} =$$

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$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} x + 1 = 2$$

Squeeze

Limits

Squeeze Theorem

Theorem. Let *f,g,h* be functions such that

$$f(x) \leq g(x) \leq h(x)$$

for all x in the “neighbourhood” of c. If limit of *f,g,h* exists on *x=c*, then we have the following.

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x) \leq \lim_{x \to c} h(x)$$

Moreover, if $\lim_{x \to c} f(x)$= $\lim_{x \to c} h(x)$ then all three limits are equals.

Example (3)

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) =$$

Observe that

- $\lim_{x \to 0} x^2 = 0$
- $\lim_{x \to 0} (-x^2) = 0$
- $-1 \leq \sin(\theta) \leq 1$ for all θ.

Example (3)

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) =$$

Example (3)

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$$

$$-x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) \leq x^2$$

$$\lim_{x\to 0}(-x^2) \leq \lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x^2}\right)\right) \leq \lim_{x\to 0}(x^2)$$

$$0 \leq \lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x^2}\right)\right) \leq 0$$

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right) =$$