Limits

Calculus 1

Contents

No Limits?

Limits

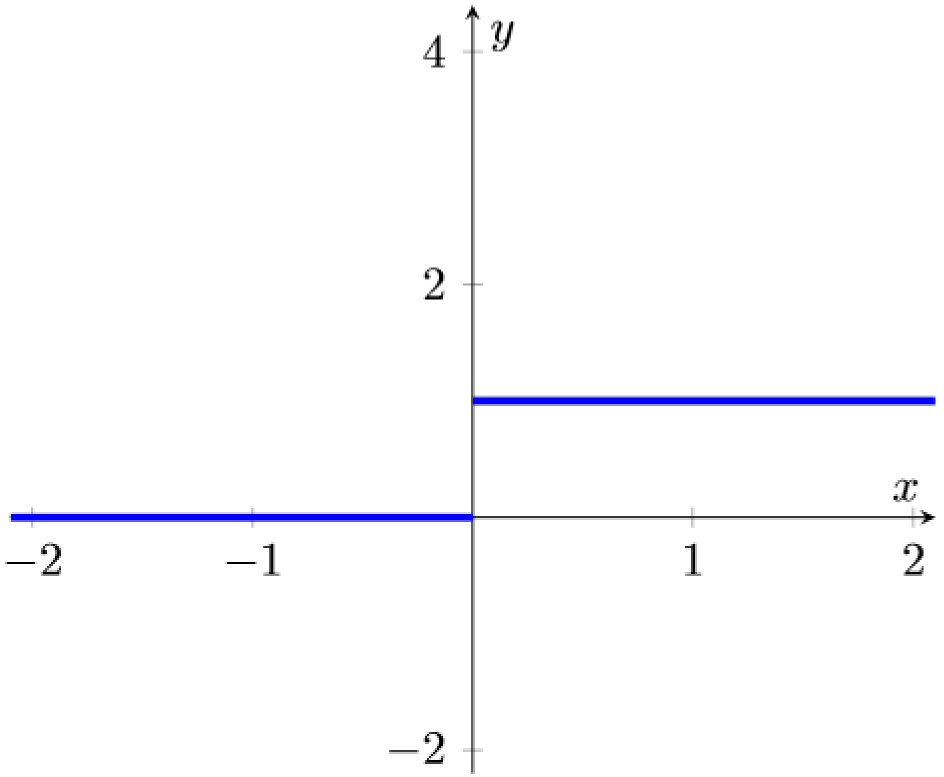
No Limits?

One Sided Limits

Limits

One Sided Limits

- Limits in the end of the intervals
- Limits on functions that has two different domain
- Limits on piecewise function



Definition

Let f be function defined on an open interval contains *c,* except possibly at c.

- If the value of $\mathrm{f}(x)$ get closer to L as $x < c$ get closer to c we say that f has a left-hand limit at c.
- If the value of $\mathrm{f}(x)$ get closer to L as $x > c$ get closer to c we say that f has a right-hand limit at c.
- We write left-hand limit and right-hand limit as

$$\lim_{x \to c^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to c^+} f(x) = L$$

respectively.

Formal Definition

Let f be function defined on an open interval contains *c,* except possibly at c. The left-hand limit of $f(x)$ as x approach *c* is said to be equals to L, denoted by

$$\lim_{x \to c^-} f(x) = L$$

if for every $\epsilon > 0$, there exist $\delta > 0$ so that every $x \in D_f$ this implication holds:

$$0 < c - x < \delta \Rightarrow 0 < |f(x) - L| < \epsilon$$

Formal Definition

Let f be function defined on an open interval contains *c,* except possibly at c. The right-hand limit of $f(x)$ as x approach *c* is said to be equals to L, denoted by

$$\lim_{x \to c^+} f(x) = L$$

if for every $\epsilon > 0$, there exist $\delta > 0$ so that every $x \in D_f$ this implication holds:

$$0 < x - c < \delta \Rightarrow 0 < |f(x) - L| < \epsilon$$

Theorem

Let f be function on interval contains c except possibly at c.

We have

$$\lim_{x\to c} f(x) = L \text{ if and only if } \lim_{x\to c^-} f(x) = L \text{ and } \lim_{x\to c^-} f(x) = L$$

Corollary

Let f be function on interval contains c except possibly at c.

If we have

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$$

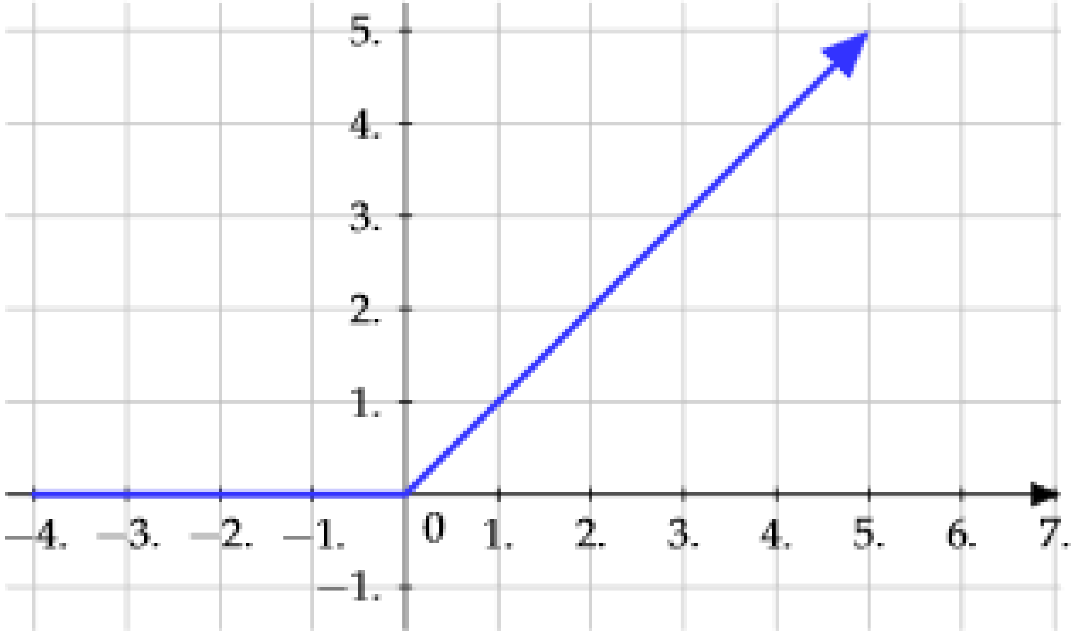
then $\lim_{x \to c} f(x)$ exist and $\lim_{x \to c} f(x) = L$.

Example

- Consider ReLU function

$$f(x) = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

- Does it have limits on *x*=0?



Example

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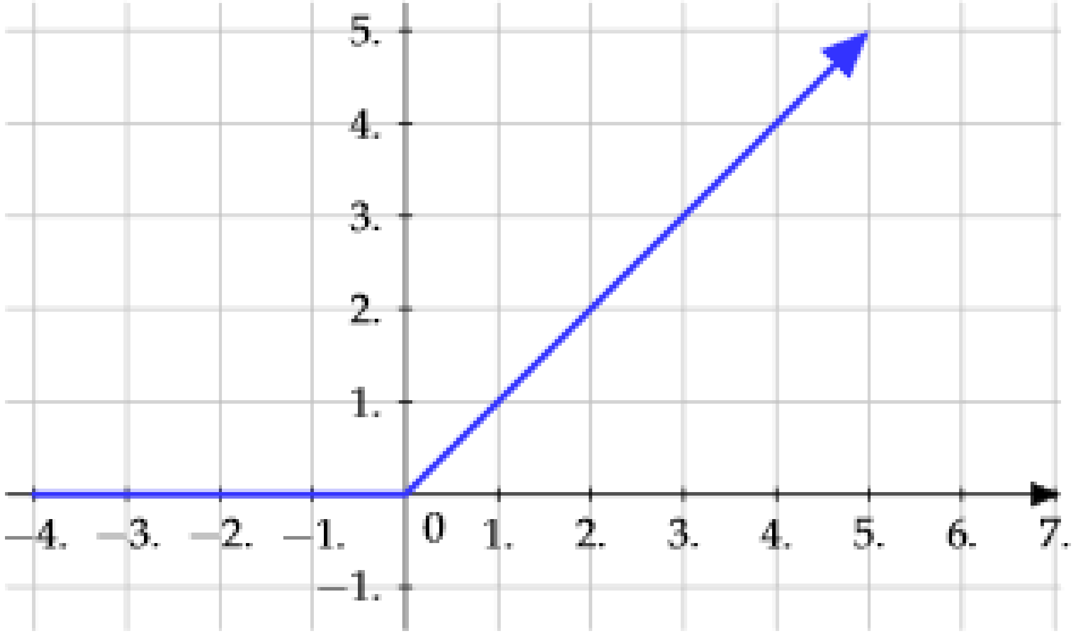
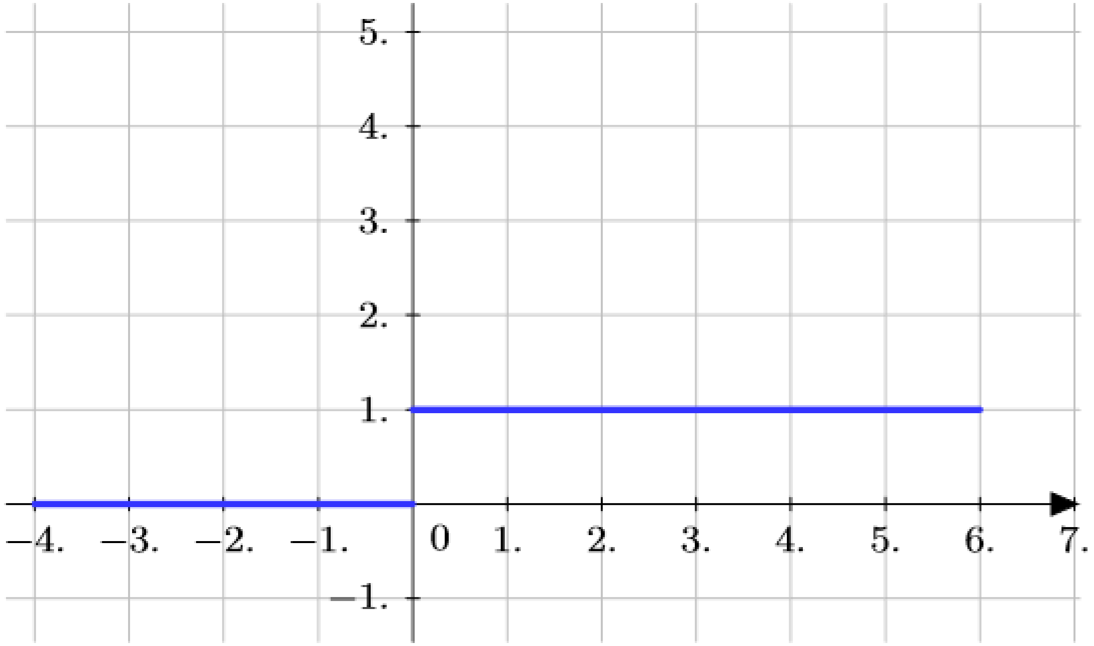


Example

- Consider step function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

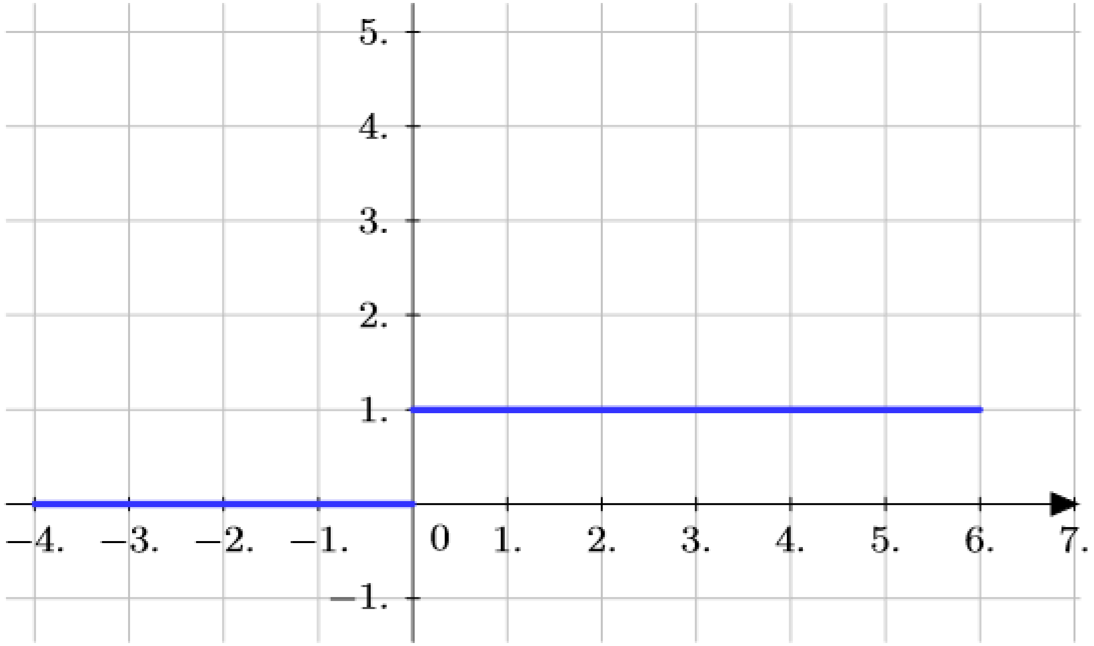
- Does it have limits on *x*=0?



Example

- Consider step function

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



- Does it have limits on *x*=0?
- $\lim_{x \to 0^-} f(x) =$
- $\lim_{x \to 0^+} f(x) =$

Example

- Consider step function

$$f(x) = \begin{cases} x^2 - 1 & x \geq 1 \\ x - 1 & x < 1 \end{cases}$$

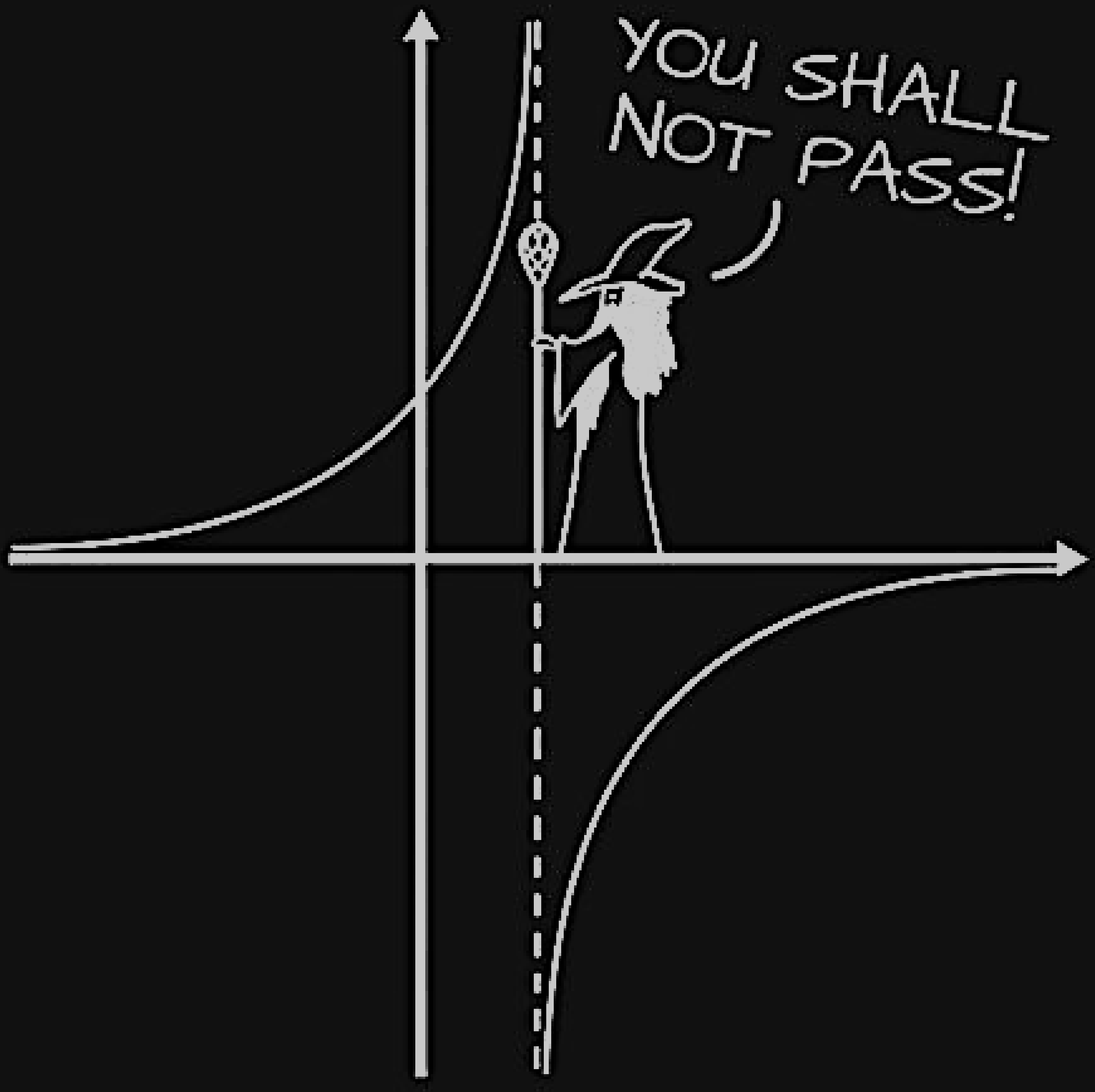
- Does it have limits on *x*=0?
- $\lim_{x \to 1^-} f(x) =$
- $\lim_{x \to 1^+} f(x) =$

QnA

Calculus 1

Infinity

Limits



Infinite limits

- Does $\lim_{x \to 0} \frac{1}{x}$ exist?
- Does $\lim_{x \to 0^-} \frac{1}{x}$ exist?
- Does $\lim_{x \to 0^+} \frac{1}{x}$ exist?

Infinite limits

- Does $\lim_{x \to 0} \frac{1}{x}$ exist?
- Does $\lim_{x \to 0^-} \frac{1}{x}$ exist?
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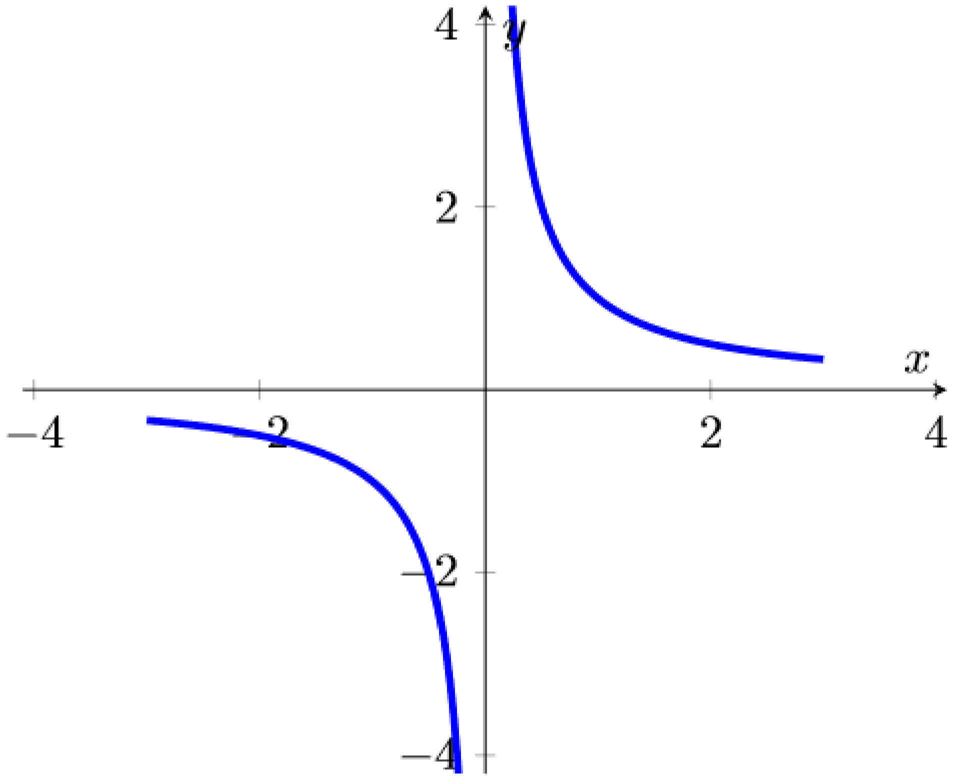
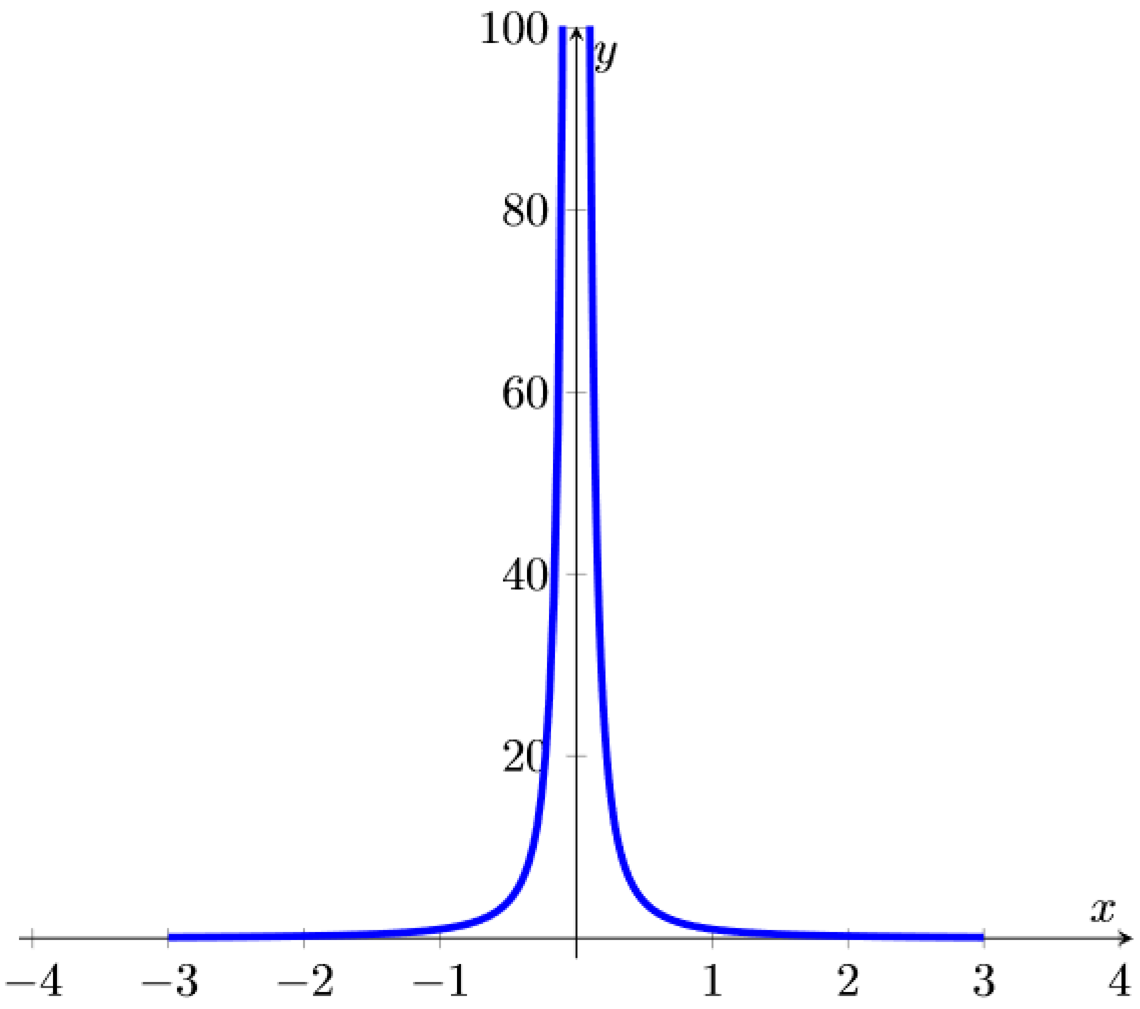


Infinite limits

- Does $\lim_{x \to 0} \frac{1}{x^2}$ exist?
- Does $\lim_{x \to 0^-} \frac{1}{x^2}$ exist?
- Does $\lim_{x \to 0^+} \frac{1}{x^2}$ exist?



Formal Definition

Let f be function defined on an open interval contains *c*, except possibly at c. The limit of $f(x)$ as x approach *c* is said to be equals to infinity, ∞, denoted by

$$\lim_{x \to c} f(x) = \infty$$

if for every $\mathrm{M} > 0$, there exist $\delta > 0$ so that every $x \in D_f$ this implication holds:

$$0 < |x - c| < \delta \Rightarrow f(x) > M$$

Formal Definition

Let f be function defined on an open interval contains *c*, except possibly at c. The limit of $f(x)$ as x approach *c* is said to be equals to minus infinity, $-\infty$, denoted by

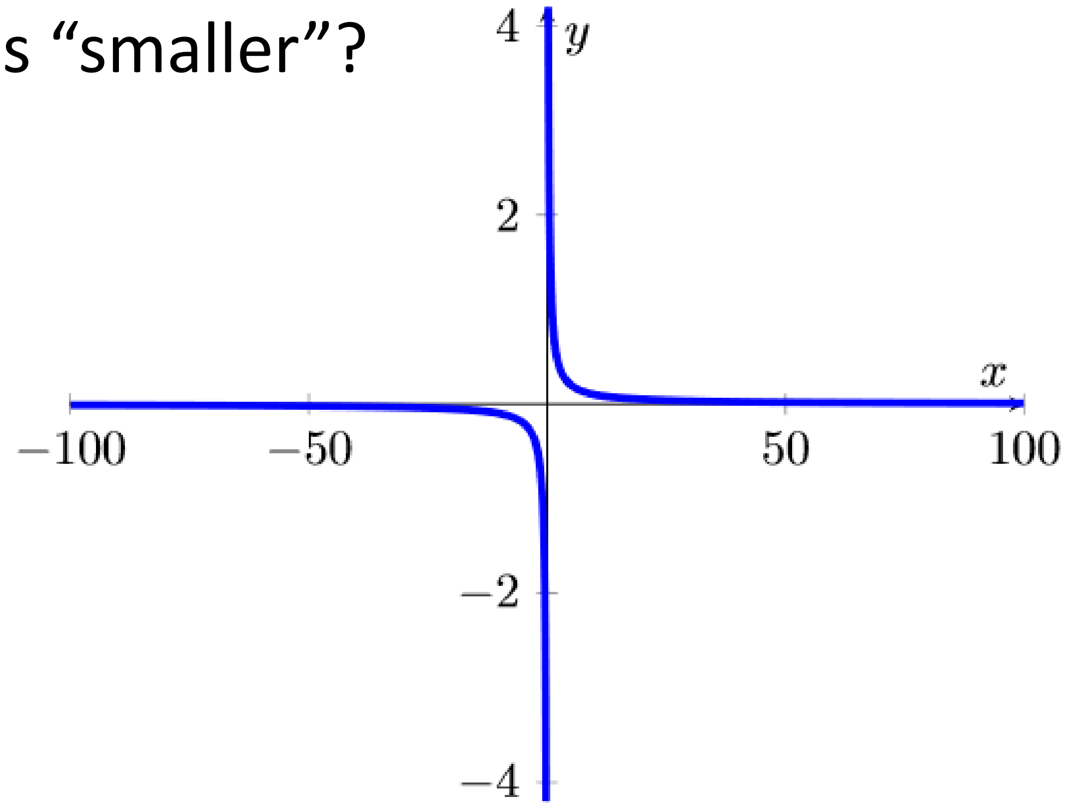
$$\lim_{x \to c} f(x) = \infty$$

if for every $\mathrm{M} > 0$, there exist $\delta > 0$ so that every $x \in D_f$ this implication holds:

$$0 < |x - c| < \delta \Rightarrow f(x) < -M$$

Limits at Infinity

- How does $f(x) = \frac{1}{x}$ behave as x goes bigger?
- How does $f(x) = \frac{1}{x}$ behave as x goes “smaller”?



Formal Definition

Let f be function defined on an open interval (k, ∞), for some k.

The limit of $f(x)$ as x approach ∞ is said to be equals to L, denoted by

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$, there exist $\mathrm{M} > 0$ so that every $x \in D_f$ this implication holds:

$$\mathrm{M} < x \Rightarrow 0 < |f(x) - L| < \epsilon$$

Formal Definition

Let f be function defined on an open interval (k, ∞), for some k.

The limit of $f(x)$ as x approach $-\infty$ is said to be equals to L, denoted by

$$\lim_{x \to -\infty} f(x) = L$$

if for every $\epsilon > 0$, there exist $\mathrm{M} > 0$ so that every $x \in D_f$ this implication holds:

$$-\mathrm{M} > x \Rightarrow 0 < |f(x) - L| < \epsilon$$

Theorem

If $r > 0$ then we have the following limits.

$$\lim_{x \to \infty} x = \infty \qquad \lim_{x \to 0^+} \frac{1}{x^r} = \infty \qquad \lim_{x \to \infty} \frac{1}{x^r} = 0$$

$$\lim_{x \to -\infty} x = -\infty \qquad \lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Theorem

If we want to compute the limits of P(x) / Q(x) one may divide by x^n where *n* is the highest power of x involved.

Example (1)

Find the following limits, if any.

$$\lim_{x \to \infty} \frac{x^2 + x + 1}{x^2 + 1} =$$

Example (1)

Find the following limits, if any.

$$\lim_{x\to\infty} \frac{x^2+x+1}{x^2+1} = \frac{\lim\limits_{x\to\infty} x^2+x+1}{\lim\limits_{x\to\infty} x^2+1}$$

$$= \frac{\lim\limits_{x\to\infty} x^2 + \lim\limits_{x\to\infty} x + \lim\limits_{x\to\infty} 1}{\lim\limits_{x\to\infty} x^2 + \lim\limits_{x\to\infty} 1}$$

Example (1)

Find the following limits, if any.

$$\lim_{x\to\infty}\frac{x^2+x+1}{x^2+1} = \lim_{x\to\infty}\frac{(x^2/x^2)+(x/x^2)+(1/x^2)}{(x^2/x^2)+(1/x^2)}$$

$$= \frac{\lim\limits_{x\to\infty}(1+\frac{1}{x}+\frac{1}{x^2})}{\lim\limits_{x\to\infty}(1+\frac{1}{x^2})}$$

$$= \frac{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x}+\lim\limits_{x\to\infty}\frac{1}{x^2}}{\lim\limits_{x\to\infty}1+\lim\limits_{x\to\infty}\frac{1}{x^2}}$$

$$=$$

Example (2)

Find the following limits, if any.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+x+1}}{x+1} =$$

Example (2)

Find the following limits, if any.

$$\lim_{x\to\infty} \frac{\sqrt{x^2+x+1}}{x+1} = \lim_{x\to\infty} \frac{\frac{1}{x}\sqrt{x^2+x+1}}{\frac{1}{x}(x+1)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{(x^2/x^2)+(x/x^2)+(1/x^2)}}{(x/x)+(1/x)}$$

$$= \frac{\sqrt{\lim\limits_{x\to\infty}(1+\frac{1}{x}+\frac{1}{x^2})}}{\lim\limits_{x\to\infty}(1+\frac{1}{x})}$$

$$=$$

QnA

Calculus 1

Limits Involving Trigonometric Functions

Limits

Limits of sin(x) and cos(x)

Theorem. For any real number c we have

- $\lim_{x \to c} \sin(x) = c$
- $\lim_{x \to c} \cos(\mathrm{x}) = c$

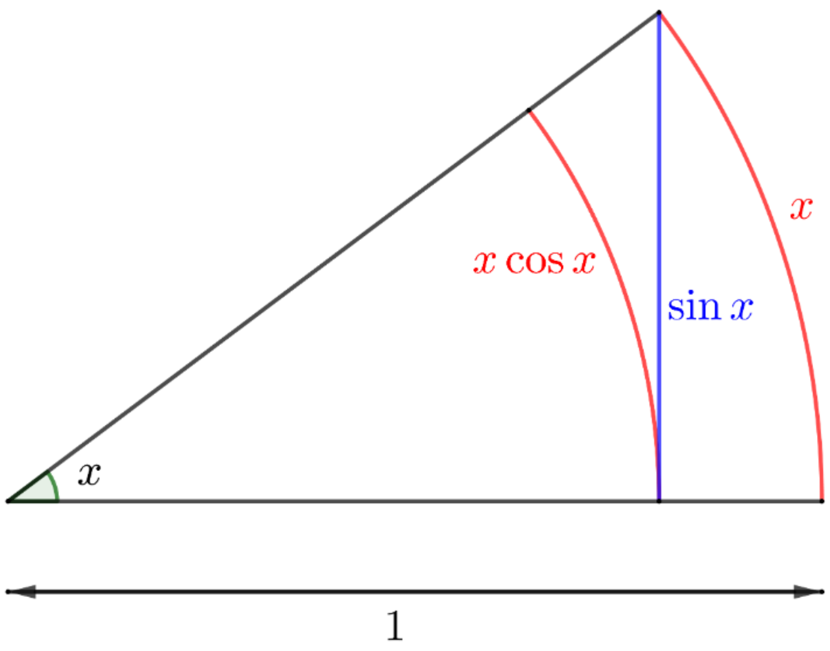
We also have that the limits of both sin(x) and cos(x), as $x \to \infty$ do not exist.

Limits of f(x) = sin(x)/x

For any x such that $0 < x < \frac{\pi}{2}$

we have

$$x \cos x \leq \sin x \leq x$$



and it means

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

Taking limits on the three sides we conclude that

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Limits of f(x) = tan(x)/x

Note that

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{1}{x}\frac{\sin x}{\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\frac{1}{\cos x} = \left(\lim_{x\to 0}\frac{\sin x}{x}\right)\left(\lim_{x\to 0}\frac{1}{\cos x}\right)$$

Hence, we have

$$\lim_{x\to 0}\frac{\tan x}{x} = 1.$$

Limits of f(x) = sin(ax)/ax

Notice that as $x \to 0$ we would expect $ax \to 0$ for any reals a. So, once we substitute $u = ax$, we would have

$$\lim_{x\to0}\frac{\sin ax}{ax} = \lim_{ax\to0}\frac{\sin ax}{ax} = \lim_{u\to0}\frac{\sin u}{u} = 1.$$

Hence $\lim_{x\to0}\frac{\sin ax}{ax} = 1.$

With the same argument, we would have $\lim_{x\to0}\frac{\tan ax}{ax} = 1.$

QnA

Calculus 1

Euler number e

Limits

Compound Interest rate 100%

Time for interest is credited	End of the year	Value
1 year	$1+1$	2
6 month	$\left(1+\frac{1}{2}\right)^{2}$	2,25
1 bulan	$\left(1+\frac{1}{12}\right)^{12}$	2,61304
1 hari	$\left(1+\frac{1}{365}\right)^{365}$	2,71456
1 jam	$\left(1+\frac{1}{24\times 365}\right)^{24\times 365}$	2,71813
1 menit	$\left(1+\frac{1}{60\times 24\times 365}\right)^{60\times 24\times 365}$	2, 71828
1 detik	$\left(1+\frac{1}{60\times 60\times 24\times 365}\right)^{60\times 60\times 24\times 365}$	2, 71828

Compound Interest rate 100%

Euler number

Theorem. The limits of the following sequences does exist.

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

and the limits called *e,* the Euler number.

Euler number

Theorem. The limits of the following sequences does exist.

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

and the limits called *e,* the Euler number.

Euler number

What does e for?

- Natural logarithm on computing.

$$\log x = \log_{10} x \text{ or } \log x = \log_e x$$

- Any exponential function $f(x) = a^x$ could be represented as

$$f(x) = e^{a \log x}$$

- Function $f(x) = e^x$ is the only function that has derivative itself.

Natural number *e* on Machine Learning

- Sigmoid function

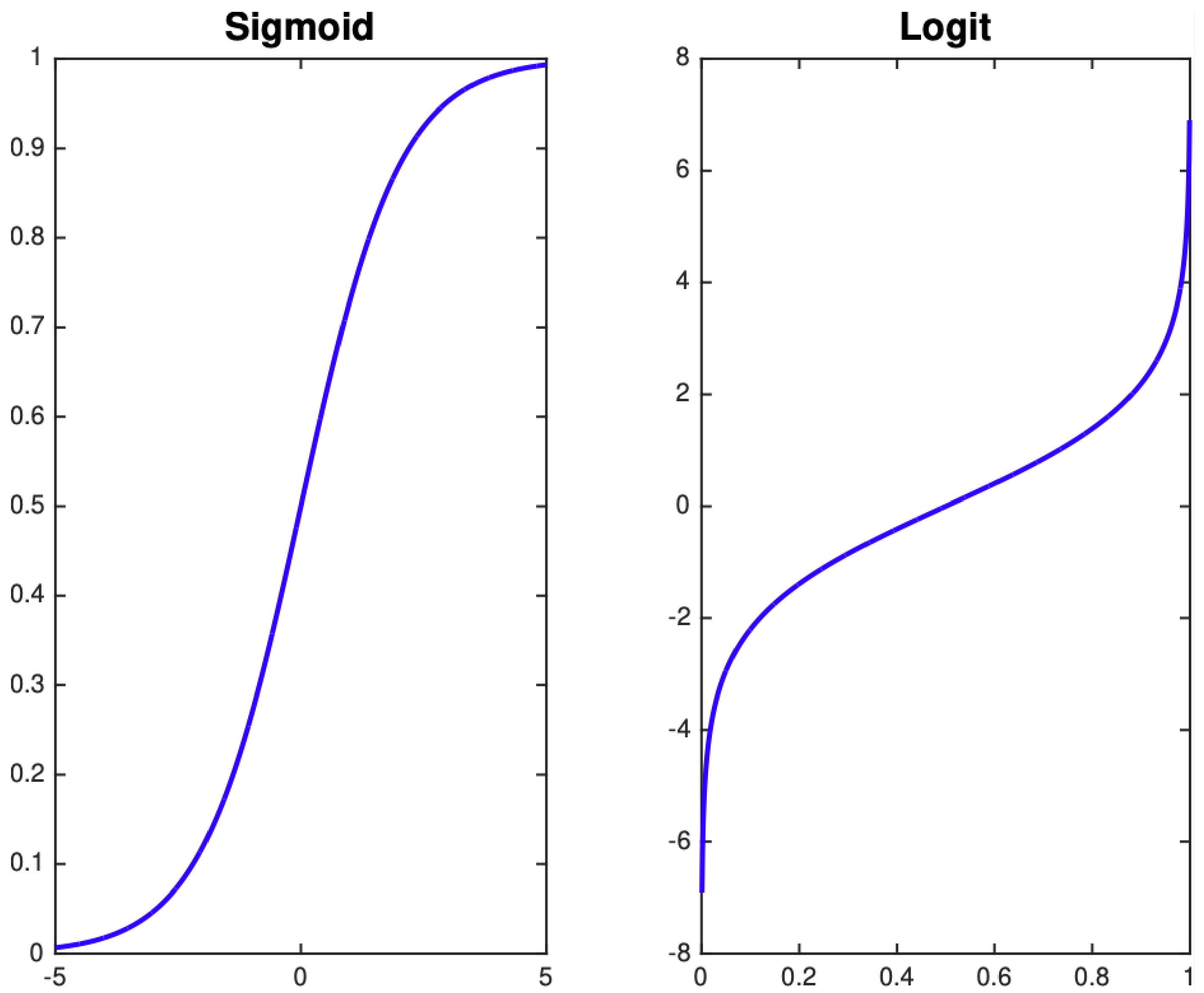
$$sigm(x) = \frac{1}{1+e^{-x}} \ ,$$

for $x \in \mathbb{R}$

- Logit

$$logit(x) = \log\frac{x}{1-x}$$

for $x \in (0,1)$



Natural number *e* on Machine Learning

- Standard *normal density*

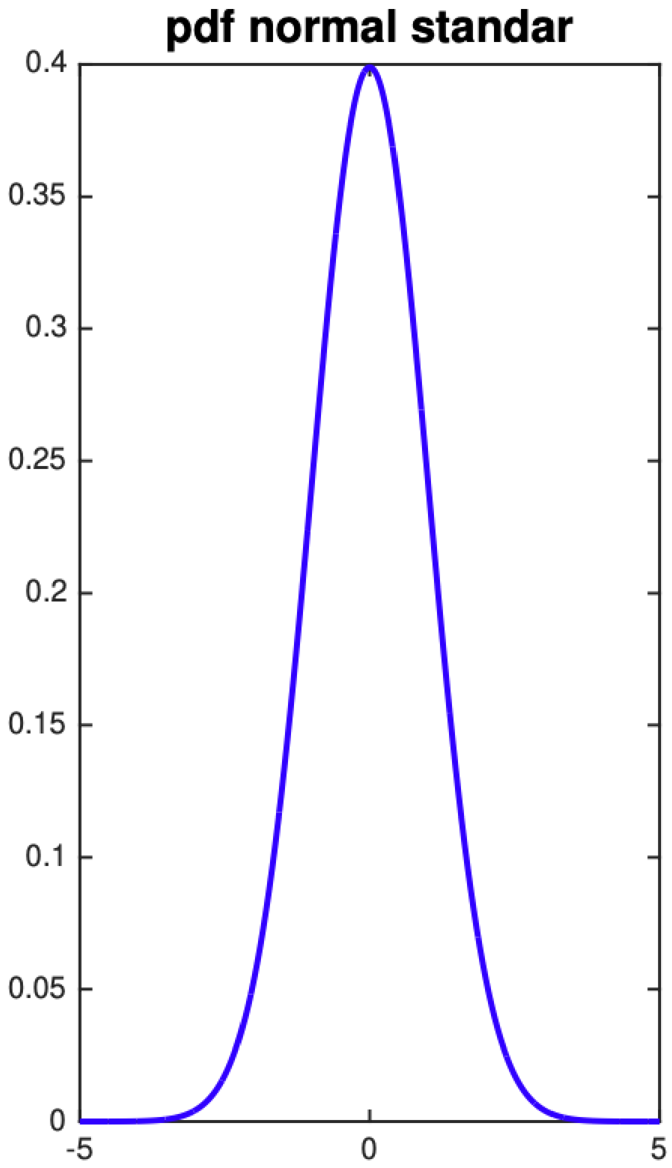
$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

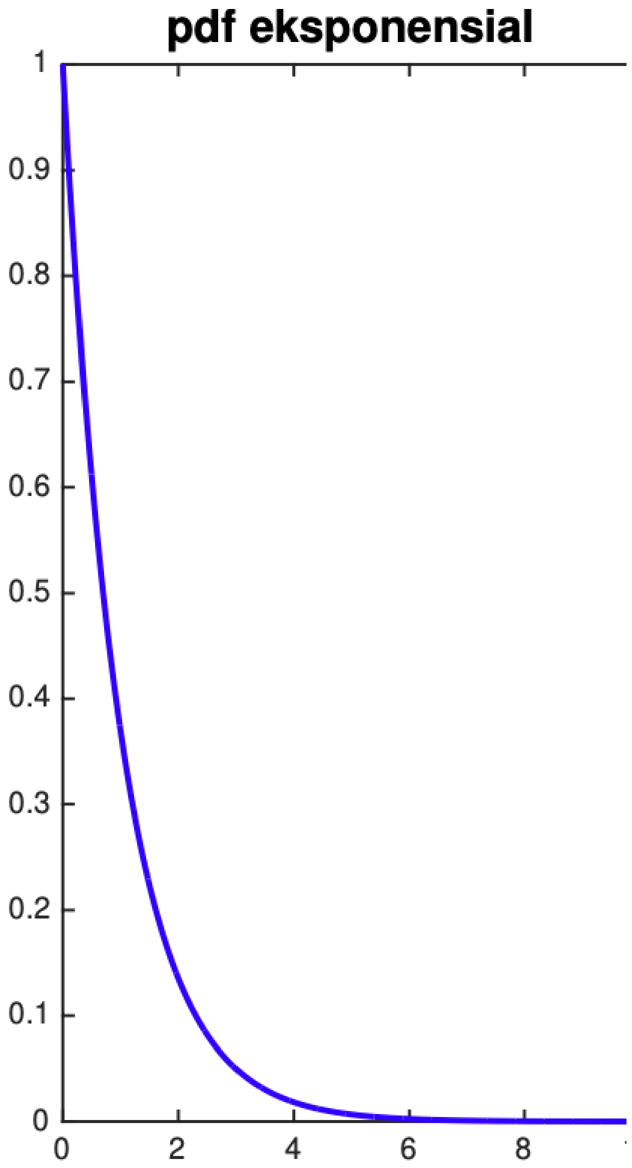
For any $x \in \mathbb{R}$.

- Exponential density

$$e(x;\lambda) = \lambda e^{-\lambda x},$$

for $x \in [0,\infty)$.





Limits involving e

Some limits involving e that one could derive from the definition.

- $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}$ (the definition)
- $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{\mathrm{e}}$ (taking $x \leftarrow -x$)
- $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \mathrm{e}$ (taking $x \leftarrow -x$)
- $\lim_{x\to 0}(1+x)^{1/x} = \mathrm{e}$ (taking $x \leftarrow \frac{1}{x}$ from prev)

Example

Find the following limits, if any.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} =$$

Example

Find the following limits, if any.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}\left(\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right)^{ab}$$

as $x\to\infty$ we have $\frac{x}{a}\to\infty$. Substitute $u=\frac{x}{a}$ and we would have

$$= \lim_{x/a\to\infty}\left(\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right)^{ab} = \lim_{u\to\infty}\left(\left(1+\frac{1}{u}\right)^{u}\right)^{ab} = e^{ab}$$

Example

Find the following limits, if any.

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}\left(\left(1+\frac{a}{x}\right)^{\frac{x}{a}}\right)^{ab}$$

as $x \to \infty$ we have $\frac{x}{a} \to \infty$.

QnA

Calculus 1

End of Session

Calculus 1